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ON LOCALLY CONFORMAL ALMOST KÄHLER MANIFOLDS

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ABSTRACT

In the first section of this note, we discuss locally conformal symplectic manifolds, which are differentiable manifolds V^{2n} endowed with a non-degenerate 2-form Ω such that $d\Omega = \omega \wedge \Omega$ for some closed form ω. Examples and several geometric properties are obtained, especially for the case when $d\Omega \neq 0$ at every point. In the second section, we discuss the case when Ω above is the fundamental form of an (almost) Hermitian manifold, i.e. the case of the locally conformal (almost) Kähler manifolds. Characterizations of such manifolds are given. Particularly, the locally conformal Kähler manifolds are almost Hermitian manifolds for which some canonically associated connection (called the Weyl connection) is almost complex. Examples of locally conformal (almost) Kähler manifolds which are not globally conformal (almost) Kähler are given. One such example is provided by the well-known Hopf manifolds.

It is well known that many particular classes of almost Hermitian manifolds have been intensively studied. Among them, almost Hermitian manifolds whose metric is globally conformal to an almost Kähler metric have been also encountered (for instance, by A. Gray [6], A. Gray and L. Vanhecke [7], etc.). But, obviously, these manifolds have the same topological properties like the almost Kähler manifolds. Therefore, it seems interesting to study almost Hermitian manifolds which are only locally conformal to an almost Kähler manifold, and it is the aim of this note to discuss briefly such manifolds[†]. More precisely, we shall give characterizations and examples of locally conformal almost Kähler manifolds and, since the main characterization follows from the symplectic geometry, we consider in the first section the locally conformal symplectic manifolds while the second section is devoted to the locally conformal almost Kähler and Kähler manifolds.

An interesting example is provided by the Hopf manifolds (see, for instance,

[†] I should like to acknowledge that my idea of considering these manifolds came during a lecture of Prof. L. Vanhecke in which globally conformal almost Kähler metrics were discussed.

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[9]), which have, as we shall see, a locally conformal Kähler metric while it is known that they admit no Kähler metric.

1. Locally conformal symplectic manifolds

Throughout this note all the manifolds and tensor fields are assumed C^∞. Let V be a 2n-dimensional connected paracompact manifold endowed with a non-degenerate 2-form Ω, i.e. (V,Ω) is an *almost symplectic manifold.* We call (V,Ω) a *locally conformal symplectic* (l.c.s.) manifold if every point $x \in V$ has an open neighbourhood U such that

$$d(e^{-\sigma}\,\Omega|_U)=0 \tag{1.1}$$

for some function $\sigma: U \to R$. If (1.1) is valid for $U = V$, (V,Ω) is *globally conformal symplectic* (g.c.s.) and if (1.1) is valid for $\sigma = \text{const.}$, (V,Ω) is a *symplectic manifold.*

The characterization of the introduced manifolds is given by:

THEOREM 1.1. (H. C. Lee [10]). *The pair (V,Ω) as above is an l.c.s. manifold iff there is a globally defined 1-form ω on V such that*

$$d\Omega = \omega \wedge \Omega, \quad d\omega = 0, \tag{1.2}$$

and (V,Ω) is g.c.s. iff, moreover, the closed form ω is exact.

PROOF. If (V,Ω) is l.c.s., (1.1) implies $d\Omega = d\sigma \wedge \Omega$, i.e. on each U there is a closed 1-form $\omega_U = d\sigma$ such that $d\Omega = \omega_U \wedge \Omega$. Clearly, on each $U \cap U'$ we have then $(\omega_U - \omega_{U'}) \wedge \Omega = 0$ and, since we cannot have $\Omega = (\omega_U - \omega_{U'}) \wedge \alpha$ because Ω is non-degenerate, it follows that $\omega_U = \omega_{U'}$. Hence, we get on V a globally defined 1-form ω verifying (1.2). Conversely, if ω exists, every $x \in V$ has a neighbourhood U where ω is exact, i.e. $\omega = d\sigma$ and (1.2) implies (1.1). The last assertion of the theorem is obvious.

It is worth remarking that the previous proof and characterization are valid for Banach manifolds too. For the finite-dimensional case, the form ω is just the one introduced by Lee [10], who also proved that for $n = 2$ the first condition (1.2) is automatically verified; we therefore call ω the *Lee form*. For $n = 1$, we clearly must take $\omega = 0$ and (V,Ω) is always symplectic. As for finite $n > 2$, it is easy to derive that the second condition (2) follows from the first [11, 15]. Finally, a characterization equivalent to (1.2) in terms of the existence of some special connections on V is derived by R. Miron in [12, 13].

COROLLARY 1.2. *For every l.c.s. manifold (V,Ω)*, the universal covering manifold $\tilde V$ *with the induced form $\tilde\Omega$ is a g.c.s manifold.*

Actually, this follows from the last assertion of Theorem 1.1 in view of the 1-connectedness of $\tilde{V}$. (Clearly, every l.c.s. structure on a 1-connected manifold is g.c.s.)

Let us also remark that the last assertion of Theorem 1.1 implies other consequences too. For instance, for the l.c.s. manifold (V, Ω) not to be g.c.s. it is necessary that $H^1(V, R) \neq 0$, or in the compact case that the first Betti number $b^1(V) \neq 0$, which implies that the fundamental group $\pi_1(V)$ is infinite.

Next, we shall give examples of l.c.s. manifolds which are not g.c.s. and which show that this class of manifolds is large enough.

First, let M^{2n-1} be a contact manifold with contact form θ, i.e. $\theta \wedge (d\theta)^{n-1} \neq 0$ [2]. Consider $V = S^1 \times M^{2n-1}$ endowed with the form $\Omega = \theta \wedge \omega + d\theta$, where ω is the *length element* of the circle S^1. Clearly, Ω is non-degenerate, ω is closed and not exact and (1.2) is verified. Hence, (V, Ω) is an l.c.s. manifold having ω as its Lee form and which is not g.c.s. Moreover, if M is compact, V is also compact. Also, let $p: V \to M$ be an arbitrary principal bundle with group S^1 over a $(2n-1)$-dimensional manifold M. Next, let θ be the form of a connection on this principal bundle and $\Psi = d\theta$ be the corresponding curvature form [8]. Then, if $\omega \wedge \theta \wedge \Psi^{n-1} \neq 0$, the form $\Omega = \theta \wedge \omega + \Psi$ defines an l.c.s. structure on V which is not g.c.s.

Then, a second type of example can be obtained as follows. Let (W^{2n-2}, χ) be a symplectic manifold, whose fundamental form χ is exact, i.e. $\chi = d\theta$ for some global 1-form θ on V and rank $(d\theta) = 2n - 2$. For example, such symplectic manifolds are the cotangent bundles of the $(n-1)$-dimensional manifolds and the tangent bundles of the $(n-1)$-dimensional Riemann manifolds (see, for instance, [1]). Next, let $p: V \to W$ be a trivial principal bundle with group T^2, the 2-dimensional torus. Then, by taking two independent invariant 1-forms on T^2, we may induce two closed and non-exact 1-forms ω and ϖ on V and we may define on V the 2-form

$$\Omega = \omega \wedge (\varpi - p^*\theta) + p^* d\theta,$$

which is obviously of rank $2n$ and satisfies (1.2). Hence (V, Ω) is an l.c.s. manifold which is not g.c.s.

Now, returning for a little while to the general l.c.s. manifolds let us remark that many interesting gometric elements may be defined for them. Thus, there will be a *canonical vector field* A defined by[†]

[†] For the values of exterior forms, products and derivatives we use the definitions of [18], also used in [6]. Other possible conventions are those of [1, 9].

$$\Omega(X, A) = \omega(X) \tag{1.3}$$

for every vector field X on V, and for which easy calculations based on (1.2) and (1.3) give

$$\omega(A) = 0, \quad L_A\omega = 0, \quad L_A\Omega = 0, \tag{1.4}$$

where L_A denotes the Lie derivative. Also, there are a lot of important linear connections, namely the *symplectic connections* ($\nabla_X\Omega = 0$) and the *conformal symplectic connections* ($\nabla_X\Omega = \alpha(X)\Omega$), which were studied by many authors: Ph. Tondeur [17], R. Miron [12, 13], V. Cruceanu [4], V. Oproiu [14].

Finally, if we introduce a Riemann metric g on the l.c.s. manifold (V, Ω) we can obtain a *second canonical vector field* B defined by

$$g(X, B) = \omega(X). \tag{1.5}$$

The following result is also worthy of mention:

PROPOSITION 1.3. *Let (V, g, Ω) be a compact Riemann l.c.s. manifold. Then, by a global conformal transformation of Ω we get on V a l.c.s. structure $\bar{\Omega}$ whose Lee form is harmonic.*

PROOF. Since V has a volume element, namely Ω^n, it is also orientable and from Hodge's decomposition theorem we have $\omega = H\omega + df$, where $H\omega$ is a harmonic 1-form and f a function $V \to R$. Then, the proposition follows by taking $\bar{\Omega} = e^{-f}\Omega$.

An interesting particular class of l.c.s. manifolds is defined by asking $d\Omega \neq 0$ at every point or equivalently $\omega \neq 0$ or $A \neq 0$ at every point. We call such manifolds *strongly nonsymplectic* (s.n.l.c.s.). If V is a compact s.n.l.c.s. manifold its Euler–Poincaré characteristic $\chi(V) = 0$. All the previously considered examples of l.c.s. manifolds were s.n.

For the remaining part of this section, (V, Ω) will be a $2n$-dimensional s.n.l.c.s. manifold and g will be an arbitrary Riemann metric on V.

We begin by remarking that, in this case, $\omega = 0$ defines on V a foliation $\mathcal{F}$ of codimension one, which we call the *canonical foliation* of V and which is transversally parallelizable in the sense of L. Conlon [3]. Clearly, B defined by (1.5) is normal to $\mathcal{F}$ and never vanishes.

Next, we resume some geometric properties of the manifold by:

THEOREM 1.4. (i) *On every Riemannian s.n.l.c.s. manifold (V, g, Ω) with Lee form ω there is a uniquely determined 1-form ϖ and a uniquely determined 2-form Ψ such that*

$$\Omega = \omega \wedge \varpi + \Psi, \tag{1.6}$$

where

$$\varpi(B) = 0, \quad \Psi(X, B) = 0, \quad \omega \wedge \varpi \wedge \Psi^{n-1} \neq 0, \quad d\Psi = \omega \wedge (\Psi + d\varpi), \tag{1.7}$$

$$\varpi(A) = 1, \quad \Psi(X, A) = 0,$$

X being an arbitrary vector field on V.

(ii) *Conversely, if on the Riemann manifold V^{2n} with metric g there are given: a closed* 1*-form ω, a* 1*-form ϖ and a* 2*-form Ψ such that, with B given by* (1.5), *the first four relations* (1.7) *hold, then* (1.6) *defines on V a unique s.n.l.c.s. structure, whose associated elements are just the given ones.*

(iii) *The leaves of the canonical foliation $\mathcal{F}$ of (V, g, Ω) have an induced almost cosymplectic structure with closed* 2*-form.*

(iv) *If V is compact, the universal covering manifold $\tilde{V}$ of V is diffeomorphic to $R \times \tilde{L}$, where $\tilde{L}$ is the universal covering manifold of an arbitrary leaf L of $\mathcal{F}$. All the leaves of $\mathcal{F}$ are diffeomorphic.*

Proof. (i) Let (V, g, Ω) be given. Then, expressing Ω by local cobases on V which include ω, we obtain local forms ϖ, Ψ such that (1.6) and the first two of the relations (1.7) hold. Moreover, these forms are uniquely defined since, if we also had

$$\Omega = \omega \wedge \varpi' + \Psi', \quad \varpi'(B) = 0, \quad \Psi'(X, B) = 0,$$

it would follow that

$$\omega \wedge (\varpi - \varpi') = \Psi' - \Psi,$$

and by evaluating these forms for the pair of arguments (X, B) one gets $\varpi = \varpi'$, $\Psi = \Psi'$. Next, it is clear that this uniqueness property assures that ϖ and Ψ are, actually, globally defined forms. As for the last four of the relations (1.7), they follow respectively from rank $\Omega = 2n$, from (1.2) and from the evaluation of (1.6) on the arguments (B, A) and (X, A) with the definition (1.3) of A.

(ii) Clearly, under the hypotheses, (1.6) defines a s.n.l.c.s. structure on V and $d\Omega = \omega \wedge \Omega$. Hence ω is the Lee form of the structure, and B defined by ω is the second canonical vector field. Hence we must have $\Omega = \omega \wedge \varpi' + \Psi'$ and just as for (i), it follows that $\varpi' = \varpi$, $\Psi' = \Psi$, which proves the assertion of the theorem.

(iii) Let $i: L \to V$ be the immersion of a generic leaf of $\mathcal{F}$ into V. Then, $i^*\Psi = \varphi(L)$ defines, in view of (1.7), a closed 2-form of rank $2n-2$ on L and $i^*\varpi = \kappa(L)$ defines a 1-form on L such that $\kappa \wedge \varphi^{n-1} \neq 0$. Hence, the anounced structure actually exists and consists of the pair of forms $(\kappa(L), \varphi(L))$. If ϖ is also a closed form the leaves have cosymplectic structures and many interesting properties can be derived [5].

(iv) The first assertion of this point follows, for instance, from the theorems of L. Conlon [3]. As for the last, if we factorize $R \times \tilde{L}$ by $id. \times \pi_1(L)$ we get for V a covering manifold of the form $R \times L$ and L covers all the leaves of $\mathcal{F}$. Since L is arbitrary, any two leaves of $\mathcal{F}$ cover each other, whence they are diffeomorphic.

2. Locally conformal almost Kähler manifolds

Let V^{2n} be an almost Hermitian manifold with the metric g and with the almost complex structure J ($J^2 = -id.$). Then

$$\Omega(X, Y) = g(X, JY) \tag{2.1}$$

defines the fundamental 2-form of V, which is non-degenerate and gives an almost symplectic structure on V. The manifold V will be called *locally (globally) conformal (almost) Kähler* (l.(g.)c.(a.)K) if Ω defines an l.(g.)c.s. structure on V. Equivalently, we may ask that every $x \in V$ has an open neighbourhood U such that for some $\sigma: U \to R$, $g' = e^{-\sigma} g|_U$ is an (almost) Kähler metric on U.

Now, the results of the previous section can be transposed to l.c.a.K. manifolds. From Theorem 1.1, we see that such manifolds are again characterized by the conditions (1.2), where ω is the Lee form. This implies the known topological consequences, and particularly, that the universal covering manifold $(\tilde{V}, \tilde{g})$ of V is a g.c.a.K. manifold, i.e. V is an almost Kähler manifold.

We can consider again the canonical vector field A defined by (1.3) or by the equivalent relation

$$g(X, A) = \omega(JX). \tag{2.2}$$

Also, we shall use g to define the second canonical vector field B, given by (1.5), and since now

$$\omega(X) = \Omega(X, A) = g(X, JA),$$

it follows that

$$B = JA, \quad A = -JB. \tag{2.3}$$

The following results are noteworthy:

PROPOSITION 2.1. *The canonical vector field A is a Killing vector field on V iff it is an infinitesimal automorphism of the almost complex structure J and in this case one has* $[A, B] = 0$.

PROOF. Using $L_A\Omega = 0$ given by (1.4) and the usual definition of the Lie derivative one gets

$$(L_Ag)(X, Y) = g(JX, [A, JY] - J[A, Y]),$$

which proves the first of the announced assertions. Next, using also $L_A\omega = 0$, one gets

$$(L_Ag)(X, B) = -g(X, [A, B])$$

which vanishes if A is Killing, and this proves the last assertion.

Obviously, of special interest are the manifolds which are l.c.K. and we shall give here an interesting characterization for this case. We remark first that the system $\{U, e^{-\sigma}g|_U\}$ is a system of local metrics on V, which are conformally related over each intersection $U \cap U'$, i.e. this is a *local metric of conformal type* in the sense of G. Tallini [16]. Next, let ∇ be the Levi–Civita connection of g and let us define

$$\tilde{\nabla}_X Y = \nabla_X Y - \tfrac{1}{2}\omega(X)Y - \tfrac{1}{2}\omega(Y)X + \tfrac{1}{2}g(X, Y)B. \tag{2.4}$$

This is easily seen to be a torsionless linear connection on V which is called the *Weyl connection* of g [16]. By simple computations one gets

$$\tilde{\nabla}_X g = \omega(X)g \tag{2.5}$$

and next, using the fact that locally $\omega = d\sigma$, $\tilde{\nabla}_X(e^{-\sigma}g) = 0$, which shows that the considered local metric has the *global connection* (2.4) [16].

Now, we can prove:

THEOREM 2.2. *The almost Hermitian manifold (V, J, g) is an l.c.K. manifold iff there is a closed 1-form ω on V such that the Weyl connection* (2.4) *be almost complex* (*i.e.* $\tilde{\nabla}_X J = 0$).

PROOF. Let V be l.c.K. and consider the Weyl connection (2.4) constructed with the Lee form ω. Consider the following classical relation, which is valid for any almost Hermitian manifold [6, 9]:

$$g((\nabla_X J)Y, Z) = \tfrac{1}{2}[d\Omega(X, JY, JZ) - d\Omega(X, Y, Z)] + \tfrac{1}{4}g(N(Y, Z), JX), \tag{2.6}$$

where N is the Nijenhuis tensor of J. In our case, $N = 0$ and we may compute the second member of (2.6) using $d\Omega = \omega \wedge \Omega$. As for the first member of (2.6), we shall use first the relation

$$(\nabla_X J)(Y) = \nabla_X (JY) - J(\nabla_X Y), \tag{2.7}$$

which is easily seen to be valid for any linear connection on V, hence for $\tilde{\nabla}$ too. Then we shall replace ∇_X from (2.4). The final result of these transformations of formula (2.6) will be

$$g((\tilde{\nabla}_X J) Y, Z) = 0, \tag{2.8}$$

which proves the necessity of the imposed condition.

Conversely, suppose we have $\tilde{\nabla}_X J = 0$. Then, since $\tilde{\nabla}$ is torsionless, it follows by a classical result [9] that the almost complex structure J is integrable. Next, using again $\tilde{\nabla}_X J = 0$, the formula (2.7) for the connection $\tilde{\nabla}$ and (2.5), we get

$$\tilde{\nabla}_X \Omega = \omega(X)\Omega. \tag{2.9}$$

Finally, from (2.9) and the known formula

$$d\Omega(X, Y, Z) = \mathfrak{S}\tilde{\nabla}_X (Y, Z), \tag{2.10}$$

where $\mathfrak{S}$ denotes the cyclic sum over X, Y, Z, it follows that $d\Omega = \omega \wedge \Omega$, i.e. V is an l.c.K. manifold.

It should also be remarked that for $n > 2$, we may not ask in advance that ω be closed, since this property follows as was shown in Section 1. Moreover, it is known that for $n > 1$, ω must be just the Lee form of Ω. Hence, we can reformulate the previous theorem in the following manner. Let (V, J, g) be an almost Hermitian manifold and ω be the Lee form of its fundamental form (which is determined by J and g). Then, take the connection (2.4) with this form ω; it is completely determined by J and g and we call it the Weyl connection of the almost Hermitian manifold. One has:

THEOREM 2.2′. *The almost Hermitian manifold* $(V^{2n}, J, g)(n > 2)$ *is an l.c.K. manifold iff its Weyl connection is almost complex. For* $n = 2$ *one must add the condition* $d\omega = 0$. *For* $n = 1$ *the manifold is always Kähler.*

It is important that this characterization is expressed only by the tensor fields g and J.

The formula (2.6) also gives another interesting result:

PROPOSITION 2.3. *The Nijenhuis tensor of any l.c.a.K. manifold satisfies the relation*

$$\mathfrak{S} g(N(Y,Z),JX)=0. \tag{2.11}$$

Proof. By the usual definition of the covariant derivative one gets

$$(\nabla_X \Omega)(Y,Z)=g(Y,(\nabla_X J)Z). \tag{2.12}$$

Using this relation to express the first member of (2.6) and $d\Omega=\omega\wedge\Omega$ to express the second member, and taking the cyclic sum over X, Y, Z, it remains just (2.11).

Next, like for the l.c.s. manifolds, it is interesting to consider l.c.a.K. manifolds which are *strongly non-Kähler* (s.n.-K.) in the sense that $d\Omega\neq 0$ (or $\omega\neq 0$ or $A\neq 0$) at every point of V, in which case one can transpose the considerations made for s.n.l.c.s. manifolds, and, especially, to consider the *canonical foliation* $\mathcal{F}$. The following theorem characterizes such manifolds:

Theorem 2.4. *On every s.n.-K.l.c.a.K. manifold V^{2n}, there is a unique closed and nowhere vanishing 1-form ω and a unique 2-form Ψ of rank $2n-2$ such that:*

$$\begin{gathered}\omega\wedge\Psi^{n-1}\neq 0; \qquad \Psi(JX,JY)=\Psi(X,Y); \quad \textit{if} \quad \omega(X)=\omega(JX)=0,\\ \Psi(JX,X)\geqq 0; \qquad d\Psi=\omega\wedge\left[\Psi+d\left(\frac{J\omega}{\|\omega\|^2}\right)\right].\end{gathered} \tag{2.13}$$

Conversely, if (ω,Ψ) with the above hypotheses are given on (V^{2n},J), there is a unique s.n.-K.l.c.a.K. metric g on the manifold, whose associated forms are the given ones.

Proof. Suppose V has the needed metric g. Then, by Theorem 1.4(i) we have (1.6) and (1.7) and by evaluating (1.6) on the arguments (JX,A) one gets

$$(J\omega)(X)=\omega(JX)=\Omega(JX,A)=\Omega(B,X)=\omega(B)\varpi(X),$$

whence, because clearly $\omega(B)=\|\omega\|^2$

$$\varpi=\frac{1}{\|\omega\|^2}J\omega, \qquad \omega=-\|\omega\|^2 J\varpi. \tag{2.14}$$

Now, the first and the last of the conditions (2.13) follow from (1.7) and using again (1.6) we have

$$\Omega(X,Y)=\frac{1}{\|\omega\|^2}(\omega(X)\omega(JY)-\omega(Y)\omega(JX))+\Psi(X,Y), \tag{2.15}$$

whence the other conditions (2.13) follow immediately.

Conversely, suppose that the forms ω and Ψ satisfy the hypotheses of the theorem and define ϖ by (2.14) and Ω by (1.6) or equivalently by (2.15). Then $\varpi \wedge \omega \wedge \Psi^{n-1} \neq 0$ since, otherwise, because ω and ϖ are obviously linearly independent, we should have $\varpi \wedge \Psi^{n-1} = 0$, which, evaluated on the arguments $(JX, JY_1, \cdots, JY_{2n-2})$ is equivalent to $(\omega \wedge \Psi^{n-1})(X, Y_1, \cdots, Y_{2n-2}) = 0$, in contradiction to (2.13). The proved relation assures that Ω is non-degenerate.

Next, if we take $g(X, Y) = \Omega(JX, Y)$, it follows from (2.15) and (2.13) that g is a Hermitian metric and that it defines an s.n.-K.l.c.a.K. structure on V.

Now, to see that ω, Ψ are the forms which the direct part of the theorem associates with g, consider the equations

$$\Psi(X, A) = 0, \quad \omega(A) = 0, \quad \varpi(A) = 1, \tag{2.16}$$

which define a unique vector field A on V. Using (2.15) we get $\Omega(X, A) = \omega(X)$, which shows that A of (2.16) is just the canonical vector field and $B = JA$ is the second canonical vector field. But, from (2.16) we get

$$\Psi(X, B) = 0, \quad \varpi(B) = 0, \tag{2.17}$$

which in view of Theorem 1.4 proves our assertion.

Clearly, for $n > 2$, we may no longer ask in advance that $d\omega = 0$, since this property automatically follows.

We also remark that we could use now parts (iii), (iv) of Theorem 1.4 in order to obtain properties of the foliation $\mathscr{F}$ and of the universal covering of V, but we shall not consider this in detail here.

In the final part of this note, we want to give some examples of l.c.a.K. and l.c.K. manifolds.

Like in Section 1, we start again with a contact manifold, but not with an arbitrary one. According to Boothby and Wang [2], if (N, Φ) is a symplectic manifold, whose fundamental form Φ defines an integral cohomology class, there is a principal S^1-bundle $\pi: M \to N$ and a connection θ on this bundle, which gives a regular contact structure on M and is such that $d\theta = \pi^*\Phi$. Hence, let us consider a $(2n-2)$-dimensional almost Kähler manifold N with the almost complex structure j and the metric γ, and whose fundamental form Φ defines an integral cohomology class of N e.g. N could be a compact Hodge manifold. Next, consider the S^1-principal bundle $\pi: M \to N$ and the connection θ mentioned above, i.e. which gives a contact structure on M. Also, let π': $V \to M$ be a trivial principal S^1-bundle over M, ω be the 1-form defined on V by the length element of the fibre and

(2.18) $$\Omega = (\pi'^*\theta)\wedge\omega + (\pi'\circ\pi)^*\Phi$$

the corresponding 2-form on V which, according to Section 1, gives an s.n.l.c.s. structure on this manifold. We shall show that Ω is the fundamental form of an almost Hermitian structure on V, thereby ending the construction of the desired example.

First, $\omega = 0$ defines the canonical foliation $\mathcal{F}$ on V and, clearly, the tangent distribution of $\mathcal{F}$ is complementary to the vertical distribution made up by the tangent lines of the fibres of V. Hence, if we denote this last distribution by $\mathcal{V}$ and the first again by $\mathcal{F}$ we have

(2.19) $$T_xV = \mathcal{F}_x \oplus \mathcal{V}_x (x \in V).$$

Since $\mathcal{V}_x = \ker \pi'_*(x)$, there is an obvious isomorphism $\alpha: \mathcal{F}_x \approx im\, \pi'_*(x) = T_{\pi'(x)}M$, which extends to vector fields and (2.19) may be written as

(2.20) $$T_xV = \alpha^{-1}(T_{\pi'(x)}M) \oplus \mathcal{V}_x.$$

Next, since θ is a connection on $\pi: M \to N$, it defines horizontal spaces on M such that

(2.21) $$T_mM = \mathcal{H}_m M \oplus \mathcal{W}_m M\ (m \in M),$$

where $\mathcal{H}$ means horizontal and $\mathcal{W}$ means vertical on M. But it is known that we have again an isomorphism $\beta: \mathcal{H}_m M \to T_{\pi(m)}N$, which also extends to vector fields. Hence (2.20), (2.21) together provide a decomposition

(2.22) $$T_xV = \alpha^{-1}\circ\beta^{-1}(T_{\pi'\circ\pi(x)}N) \oplus \alpha^{-1}(\mathcal{W}_{\pi'(x)}M) \oplus \mathcal{V}_x$$

which yields corresponding decompositions of the vector fields on V.

Namely, let us denote by P the vector field on V whose trajectories are the fibres of π' and by Q the vector field on M whose trajectories are the fibres of π. Then every vector field X on V has a unique decomposition of the form

(2.23) $$X = X_1 + \lambda\alpha^{-1}(Q) + \mu P,$$

where X_1 belongs to the first term of the decomposition (2.22).

Now, we can define an operator J acting on X of (2.23) by the formula

(2.24) $$JX = \alpha^{-1}\circ\beta^{-1}\circ j\circ\beta\circ\alpha(X_1) + \mu\alpha^{-1}(Q) - \lambda P,$$

where j is the almost complex structure of N and it is obvious that $J^2 = -id.$, i.e. J defines an almost complex structure on V.

Next, if we take also the vector field $Y = Y_1 + \lambda'\alpha^{-1}(Q) + \mu' P$, an easy computation gives in view of (2.18)

$$\Omega(X, Y) = (\lambda\mu' - \mu\lambda') + \Phi(X_1, Y_1) \tag{2.25}$$

and since Φ is the fundamental form of the metric γ it follows that

$$\Omega(JX, JY) = \Omega(X, Y). \tag{2.26}$$

Hence

$$g(X, Y) = \Omega(JX, Y) = (\lambda\lambda' + \mu\mu') + \gamma(X_1, Y_1) \tag{2.27}$$

is an almost Hermitian metric on V whose fundamental form is Ω and we see that V is an l.c.a.K. manifold.

Moreover, ω never vanishes, hence V is s.n.-K. and if we start with a compact manifold N, V is compact too.

For a first example of an l.c.K. manifold let us proceed as follows. Consider the complex analytic manifold $V = T^1_c \times C^{n-1}$, where T^1_c is the complex 1-dimensional torus and C is the complex line. Denote by w and respectively $z^1, \cdots, z^{n-1}$ the complex cartesian coordinates on T^1_c and C^{n-1} and define on V the Hermitian metric

$$ds^2 = \left(dw - \sum_{i=1}^{n-1} \bar{z}^i dz^i\right) \otimes \left(d\bar{w} - \sum_{i=1}^{n-1} z^i d\bar{z}^i\right) + \sum_{i=1}^{n-1} dz^i \otimes d\bar{z}^i. \tag{2.28}$$

The corresponding fundamental form is

$$\Omega = -\sqrt{-1}\left\{\left(dw - \sum_{i=1}^{n-1} \bar{z}^i dz^i\right) \wedge \left(d\bar{w} - \sum_{i=1}^{n-1} z^i d\bar{z}^i\right)\right\} + \sum_{i=1}^{n-1} dz^i \wedge d\bar{z}^i \tag{2.29}$$

and it follows at once that

$$d\Omega = \omega \wedge \Omega, \tag{2.30}$$

where

$$\omega = dw + d\bar{w} - \sum_{i=1}^{n-1} (\bar{z}^i dz^i + z^i d\bar{z}^i), \tag{2.31}$$

which is obviously closed but not exact and, also, never vanishes.

We have thus an example of an s.n.-K.l.c.K. manifold which is not g.c.K. However, the constructed manifold also admits a Kähler metric since it is the product of two Kähler manifolds.

A more interesting example is that of the *Hopf manifolds*, which are defined as quotients $H = (C^n \backslash \{0\})/\Delta_\lambda$, where Δ_λ is the cyclic group generated by the transformation $z \mapsto \lambda z$ ($z \in C^n \backslash \{0\}$), λ being any non-zero complex number with

$|\lambda| \neq 1$. It is known (see, for instance, [9]) that H is diffeomorphic with $S^1 \times S^{2n-1}$, hence it is a compact complex analytic manifold.

Consider on $C^n \setminus \{0\}$ the Hermitian metric

$$ds^2 = \frac{\sum_{i=1}^{n} dz^i \otimes d\bar{z}^i}{\sum_{j=1}^{n} z^j \bar{z}^j}. \tag{2.32}$$

This metric is obviously invariant by Δ_λ, hence it induces a Hermitian metric on H and it is clear from (2.32) that this is an l.c.K. metric. For $n > 1$, the obtained metric cannot be g.c.K. since it is known [9] that H admits no Kähler metric. The fundamental form of the metric (2.32) is

$$\Omega = -\sqrt{-1}\left(1 \Big/ \sum_{j=1}^{n} z^j \bar{z}^j\right) \sum_{i=1}^{n} dz^i \wedge d\bar{z}^i, \tag{2.33}$$

whence

$$d\Omega = \omega \wedge \Omega,$$

where the Lee form ω is given by

$$\omega = -\frac{\sum_{i=1}^{n} (\bar{z}^i dz^i + z^i d\bar{z}^i)}{\sum_{i=1}^{n} z^i \bar{z}^i} \tag{2.34}$$

which is closed, not exact and never vanishes, i.e. H is also s.n.-K.

The interest of this example consists in the already mentioned facts that H is compact and admits no Kähler metric. We see thus that the topology of the l.c.K. manifolds may be quite different from that of the Kähler manifolds.

Note added in proof. A theorem related to Theorem 2.2 can be found in Gh. Atanasiu, C. R. Acad. Paris **278** A (1974), 501–504.

References

1. R. L. Bishop and S. I. Goldberg, *Tensor Analysis on Manifolds*, Macmillan Co., New York, 1968.

2. W. M. Boothby and H. C. Wang, *On contact manifolds*, Ann. of Math. **68** (1958), 721–734.

3. L. Conlon, *Transversally parallelizable foliations of codimension two*, Trans. Amer. Math. Soc. **194** (1974), 79–102.

4. V. Cruceanu, *Connexions compatibles avec certaines structures sur un fibré vectoriel banachique*, Czechosl. Math. J. **24** (1974), 126–142.

5. A. Fujimoto and H. Mutō, *On cosymplectic manifolds*, Tensor **28** (1974), 43–52.

6. A. Gray, *Some examples of almost Hermitian manifolds*, Illinois J. Math. **10** (1966), 353–366.

7. A. Gray and L. Vanhecke, *Almost Hermitian manifolds with constant holomorphic sectional curvature*, Preprint, 1975.

8. S. Kobayashi, *Principal fibre bundles with the 1-dimensional toroidal group*, Tôhoku Math. J. **8** (1956), 29–45.

9. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. II, Intersc. Publ., New York, 1969.

10. H. C. Lee, *A kind of even-dimensional differential geometry and its application to exterior calculus*, Amer. J. Math. **65** (1943), 433–438.

11. P. Libermann, *Sur les automorphismes infinitesimaux des structures symplectiques et des structures de contact*, Coll. Géom. Diff. Globale, Bruxelles, 1958, pp. 37–59.

12. R. Miron, *Connexions compatibles aux structures conformes presque symplectiques*, C. R. Acad. Sci. Paris, **265** (1967), 685–687.

13. R. Miron, *Espaces à structure conforme presque symplectique*, Colloq. Math. **26** (1972), 207–215.

14. V. Oproiu, *Some remarks on the conformal almost symplectic connections*, An. Şti. Univ. "Al. I. Cuza" Iaşi Sect. I a Mat. **15** (1969), 151–157.

15. W. Slebodzinski, *Exterior Forms and their Applications*, PWN, Warszawa, 1970.

16. G. Tallini, *Metriche locali dotate di una connessione globale su una varieta differentiabile*, Period. Mat. **46** (1968), 340–358.

17. Ph. Tondeur, *Affine Zusammenhänge auf Mannifaltigkeiten mit fastsymplektischer Struktur*, Comment. Math. Helv. **36** (1961), 234–244.

18. I. Vaisman, *Cohomology and Differential Forms*, M. Dekker Inc., New York, 1973.

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